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AUTHOR(S):

HATANO, Kaoru

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Bessel capacity of symmetric generalized Cantor sets

Kaoru HATANO (Shimane Univ.)
秦野 薫

§1. Introduction

In [10] M. Ohtsuka obtained a necessary and sufficient condition for a symmetric generalized Cantor set to be of zero α (or logarithmic)-capacity. In the non-linear potential theory metric properties of sets of zero Bessel capacity were investigated and the Bessel capacity of Cantor sets of special type was estimated in [8; §7]. In order to explain their results, let us recall the definitions of Bessel capacity and symmetric generalized Cantor sets.

Let $g_\alpha = g_\alpha(x)$ be the Bessel kernel of order α, $0 < \alpha < \infty$, on the n-dimensional Euclidean space $R^n (n \geq 1)$, whose Fourier transform is $(1 + |\xi|^2)^{-\alpha/2}$, and $h_\alpha = h_\alpha(x) = |x|^{\alpha-n}$ be the Riesz kernel, $0 < \alpha < n$. The Bessel capacity $B_{\alpha,p}$ is defined as follows: For a set $A \subset R^n$,

$$B_{\alpha,p}(A) = \inf \int f(x)^p dx,$$

the infimum being taken over all functions $f \in L_p^+$ such that

$$g_\alpha * f(x) \geq 1 \text{ for all } x \in A.$$

For $R_{\alpha,p}(A)$, we just replace g_α by h_α. We shall always assume that $1 < p < \infty$ and $0 < \alpha p \leq n$.

Let $\{k_j\}_{j=1}^{\infty}$ be a sequence of integers and $\{\ell_j\}_{j=0}^{\infty}$ be a sequence of positive numbers such that $k_j \geqq 2$ and $k_{j+1}\ell_{j+1} < \ell_j$ $(j \geqq 0)$. Let $\delta_{j+1} = (\ell_j - k_{j+1}\ell_{j+1})/(k_{j+1} - 1)$ $(j = 0, 1, \ldots)$. Let I be a closed interval of length ℓ_0 in R^1. In the first step, we remove from I (k_1-1) open intervals each of the same length δ_1 so that k_1 closed intervals $I_i^{(1)}$ $(i = 1, \ldots, k_1)$ each of length ℓ_1 remain. Set $E^{(1)} = \bigcup_{i=1}^{k_1} I_i^{(1)}$. Next in the second step, we remove from each $I_i^{(1)}$ $(k_2 - 1)$ open intervals each of the same length δ_2 so that k_2 closed intervals $I_{i,j}^{(2)}$ $(j = 1, \ldots, k_2)$ each of length ℓ_2 remain. We set $E^{(2)} = \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{k_2} I_{i,j}^{(2)}$. We continue this process and obtain $E^{(j)}$, $j \geqq 1$. We define $E = \bigcap_{j=1}^{\infty} E_n^{(j)}$, where the set $E_n^{(j)} = E^{(j)} \times \cdots \times E^{(j)}$ is the product set of n $E^{(j)}$'s in R^n. We call the set E the n-dimensional symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$.

The Cantor set E considered by Maz'ya and Khavin [8] is the one constructed as above with $k_j = 2$ for all $j \geqq 1$. For such a Cantor set E, they proved the following theorem.

Theorem A. If $\alpha p < n$, then

$$B_{\alpha,p}(E) = 0 \text{ is equivalent to } \sum_{j=1}^{\infty} 2^{-jn/(p-1)} \ell_j^{(\alpha p-n)/(p-1)} = \infty$$

and if $\alpha p = n$, then

$$B_{\alpha,p}(E) = 0 \text{ is equivalent to } \sum_{j=1}^{\infty} 2^{-jn/(p-1)} (-\log \ell_j) = \infty .$$

In [5] we obtain upper and lower estimates for the Bessel capacity of symmetric generalized Cantor sets. Namely, we have

Theorem. Let E be the n-dimensional symmetric generalized

Cantor set constructed by the system $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$ with $\ell_0 \leqq 1$. If $\alpha p < n$, then

$$C^{-1}\{\ell_0^{(\alpha p-n)/(p-1)} + \sum_{j=1}^{\infty}(k_1 \cdots k_j)^{-n/(p-1)}\ell_j^{(\alpha p-n)/(p-1)}\}^{1-p}$$

$$\leqq B_{\alpha,p}(E) \leqq C\{\sum_{j=1}^{\infty}(k_1 \cdots k_j)^{-n/(p-1)}\ell_j^{(\alpha p-n)/(p-1)}\}^{1-p}$$

and if $\alpha p = n$, then

$$C^{-1}\{1 + (-\log \ell_0) + \sum_{j=1}^{\infty}(k_1 \cdots k_j)^{-n/(p-1)}(-\log \ell_j)\}^{1-p}$$

$$\leqq B_{\alpha,p}(E) \leqq C\{\sum_{j=1}^{\infty}(k_1 \cdots k_j)^{-n/(p-1)}(-\log \ell_j)\}^{1-p},$$

where the number C ($\geqq 1$) depends only on n, p and α.

Remark. (i) If the condition $\ell_0 \leqq 1$ is dropped, then the assertion is still valid for the case $\alpha p < n$ but in this case the constant C also depends on ℓ_0. (ii) In case $\alpha p < n$, since $R_{\alpha,p}(A)$ is comparable to $B_{\alpha,p}(A)$ whenever diam $A \leqq r_0 (< \infty)$ and $R_{\alpha,p}(rA) = r^{n-\alpha p}R_{\alpha,p}(A)$, where $rA = \{rx;\ x \in A\}$ for $r > 0$, the above result holds replaced $B_{\alpha,p}$ by $R_{\alpha,p}$ for all ℓ_0 (> 0). (iii) In case $p = 2$, this theorem is a refinement of Ohtsuka's result in [10], because if $0 < 2\alpha < n$, then $R_{\alpha,2}$ is comparable to $C_{2\alpha}$, where $C_{2\alpha}$ denotes the Riesz capacity corresponding to the Riesz kernel $|x|^{2\alpha-n}$ and if $2\alpha = n$, then $B_{\alpha,2}(A)$ is comparable to the logarithmic capacity of A, provided diam $A \leq r_0$ (< 1). Clearly, Theorem A is a corollary of this theorem.

In §2 and §3 we shall give an outline of the proof of our theorem. As an application of our estimates in §4 we construct a set which belongs to the (β,q)-fine topology $\tau_{\beta,q}$ but not to the (α,p)-fine topology $\tau_{\alpha,p}$, provided either $0 < \beta q < \alpha p < n$ or $0 < \beta q = \alpha p < n$ and $q > p$ or $0 < \beta q < \alpha p = n$ or $\beta q = \alpha p = n$ and $q > p$, and give its brief proof. (Inclusion relations among these fine topologies have been obtained in [3; Theorem B].)

§2. The upper estimate

In this section we obtain the upper estimate. In the sequel, for simplicity, let $a = 1/(p-1)$ and $d = n - \alpha p$. We use the following theorem obtained by Maz'ya and Khavin which is a generalization of a Carleson's theorem ([4; §1V, Theorem 2]).

Theorem B ([8; Theorem 7.3]). Let A be a Borel set in R^n with diameter $\leq n^{1/2}$ and for $r > 0$, let $\mathcal{A}(r)$ be the minimum number of closed balls with radii $\leq r$ which cover A. Then

$$B_{\alpha,p}(A) \leq C\{\int_0^{n^{1/2}} (r^d \mathcal{A}(r))^{-a} r^{-1} dr\}^{1-p},$$

where C depends only on n, p and α.

Remark. In this theorem we can replace the above $\mathcal{A}(r)$ by a measurable function $\tilde{\mathcal{A}}(r)$, where A is covered by at most $\tilde{\mathcal{A}}(r)$ union of closed balls with radii $\leq r$.

Now, let A = E. Then $\mathcal{A}(r) \leq (k_1 \ldots k_{j+1})^n$ for $t_{j+1} \leq r < t_j$ ($j = 0, 1, \ldots$), where $t_j = n^{1/2}\ell_j/2$, because $E_n^{(j+1)}$ can be covered by $(k_1 \ldots k_{j+1})^n$ closed balls with radii t_{j+1}.

In the case where $\alpha p < n$, by Theorem B we can obtain

$$B_{\alpha,p}(E) \leq C\{\sum_{j=1}^{\infty}(k_1 \ldots k_j)^{-an}\ell_j^{-ad}\}^{1-p},$$

where the constant C depends only on n, p and α, and in the sequel the symbol C stands for a constant ≥ 1, whose value may vary from a line to the next. In the case where $\alpha p = n$, similarily we obtain

$$B_{\alpha,p}(E) \leqq C\{\sum_{j=1}^{\infty}(k_1\cdots k_j)^{-an}(-\log \ell_j)\}^{1-p}.$$

Thus the estimate from the above is proved.

§3. The lower estimate

To obtain the lower estimate, we may assume that $\sum_{j=1}^{\infty}(k_1\cdots k_j)^{-an}\ell_j^{-ad} < \infty$ for $\alpha p < n$ and $\sum_{j=1}^{\infty}(k_1\cdots k_j)^{-an}(-\log \ell_j) < \infty$ for $\alpha p = n$. For a Borel set A in R^n, we consider another capacity $\tilde{b}_{\alpha,p}$ defined by

$$\tilde{b}_{\alpha,p}(A) = \sup \nu(R^n),$$

where the supremum is taken over all non-negative measures ν such that $\nu(R^n \setminus A) = 0$ and $\int W_{\alpha,p}^{\nu}(x)d\nu(x) \leqq 1$. Here $B(x, r)$ denotes the open ball with center at x and radius r and

$$W_{\alpha,p}^{\nu}(x) = \int_0^1 \{r^{-d}\nu(B(x, r))\}^a r^{-1}dr.$$

Then it follows from [6; Theorem 1] (and also see [1] and [2]) that there exists a positive number C (≥ 1) such that

$$(1) \qquad C^{-1}\tilde{b}_{\alpha,p}^{p}(A) \leqq B_{\alpha,p}(A) \leqq C\tilde{b}_{\alpha,p}^{p}(A)$$

for every Borel set $A \subset R^n$.

The following lemma can be proved by using Fatou's lemma and [7; Introduction, Corollary 1 of Lemma 0.1].

Lemma. If non-negative measures μ_j converge vaguely to μ as $j \to \infty$, then for every $x \in R^n$

$$\liminf_{j\to\infty} W^{\mu_j}_{\alpha,p}(x) \geqq W^{\mu}_{\alpha,p}(x).$$

Let $\mu_j = (k_1 \dots k_j)^{-n} \ell_j^{-n} \chi_{E_n^{(j)}}\, dx$ on R^n for $j = 1, 2, \dots,$

where χ_A denotes the characteristic function of A and dx means the n-dimensional Lebesgue measure. Then $\mu_j(R^n) = 1$ and for $x \in E_n^{(j)}$ we obtain

$$(2) \qquad \mu_j(B(x, r)) \leqq \begin{cases} C(k_1 \dots k_j)^{-n} \ell_j^{-n} r^n, & 0 < r \leqq \ell_j, \\ C(k_1 \dots k_q)^{-n} s^n, & r_{q,s} \leqq r < r_{q,s+1} \end{cases}$$

$$(1 \leqq s \leqq k_q - 1,\ 1 \leqq q \leqq j),$$

where $r_{q,s} = s\ell_q + (s-1)\delta_q$, since for $r_{q,s} \leqq r < r_{q,s+1}$ $(1 \leqq s \leqq k_q - 1$ and $1 \leqq q \leqq j)$, the number of cubes composing the set $E_n^{(j)}$ which meet $B(x, r)$ is at most $(6s)^n (k_{q+1} \dots k_j)^n$.

First, we assume $\alpha p < n$ and estimate $W^{\mu_j}_{\alpha,p}$ on E. For $x \in$ E, by virtue of (2) we have

$$(3) \qquad W^{\mu_j}_{\alpha,p}(x) \leqq C\{\ell_0^{-ad} + \textstyle\sum_{q=1}^{\infty} (k_1 \dots k_q)^{-an} \ell_q^{-ad}\}.$$

Note that by our assumption the right side of (3) is convergent. From the sequence $\{\mu_j\}$ we can extract a subsequence which converges vaguely to some measure μ with support in E and $\mu(R^n) = 1$. Hence by the Lemma

$$(4) \qquad W^{\mu}_{\alpha,p}(x) \leqq C\{\ell_0^{-ad} + \textstyle\sum_{j=1}^{\infty} (k_1 \dots k_j)^{-an} \ell_j^{-ad}\}$$

for every $x \in E$. Since $\int W^{c\mu}_{\alpha,p}(x)\,d(c\mu)(x) = c^{ap}\int W^{\mu}_{\alpha,p}(x)\,d\mu(x)$ for $c > 0$, it follows from (4) that

$$\widetilde{b}_{\alpha,p}(E) \geqq C^{-1}\{\ell_0^{-ad} + \sum_{j=1}^{\infty}(k_1\ldots k_j)^{-an}\ell_j^{-ad}\}^{(1-p)/p}.$$

Thus on account of (1) we obtain the desired lower estimate in case $\alpha p < n$.

Next, we assume that $\alpha p = n$. For $x \in E$ we can obtain

$$W^{\mu_j}_{\alpha,p}(x) \leqq C\{1 + (-\log \ell_0) + \sum_{q=1}^{\infty}(k_1\ldots k_q)^{-an}(-\log \ell_q).$$

Hence by an argument similar to the above, we can prove the desired result. Thus the lower estimate is obtained.

Remark. We can obtain an integral estimate of the Bessel capacity of symmetric generalized Cantor sets. Let $\mathcal{A}(r) = (k_1 \ldots k_{j+1})^n$ for $t_{j+1} \leqq r < t_j$ $(j = 0, 1, \ldots)$ (for the definition of t_j see §2). Then we easily prove

$$C^{-1}\int_0^{n^{1/2}}(r^d\mathcal{A}(r))^{-a}r^{-1}dr \leqq \sum_{j=1}^{\infty}(k_1\ldots k_j)^{-na}\ell_j^{-ad}$$
$$\leqq C\int_0^{n^{1/2}}(r^d\mathcal{A}(r))^{-a}r^{-1}dr$$

and hence

$$C^{-1}\{\ell_0^{-ad} + \int_0^{n^{1/2}}(r^d\mathcal{A}(r))^{-a}r^{-1}dr\}^{1-p} \leqq B_{\alpha,p}(E)$$
$$\leqq C\{\int_0^{n^{1/2}}(r^d\mathcal{A}(r))^{-a}r^{-1}dr\}^{1-p}.$$

§4. Application

Following N. G. Meyers [9], we shall say that a set E is (α,p)-thin at $x \in R^n$ if

$$\int_0^1 \{r^{-d}B_{\alpha,p}(E \cap B(x,r))\}^a r^{-1}dr < \infty .$$

We define the (α,p)-fine topology $\mathcal{T}_{\alpha,p}$ (see, e.g.[3]) to be the collection of all sets $H \subset R^n$ such that $R^n \setminus H$ is (α,p)-thin at every point of H. In this section we construct sets stated in the introduction by using the estimate of Bessel capacity of Cantor sets.

Proposition. Assume that (i) $0 < \beta q < \alpha p < n$ or (ii) $0 < \beta q = \alpha p < n$ and $q > p$ or (iii) $0 < \beta q < \alpha p = n$ or (iv) $\beta q = \alpha p = n$ and $q > p$. Then there exists a generalized Cantor set E such that $(R^n \setminus E) \cup \{x_0\} \in \mathcal{T}_{\beta,q} \setminus \mathcal{T}_{\alpha,p}$, where $x_0 \in E$.

To prove the proposition, we construct a Cantor set of zero $B_{\beta,q}$-capacity which is not (α,p)-thin at each of its points. In case (i), (ii) or (iii) let $k_j = 2$ for $j \geq 1$ and let $\ell_j = \{2^{-n(j+j_0)}(j + j_0)^{q-1}\}^{1/(n-\beta q)}$ for $j \geq 0$, where j_0 is so chosen that $2\ell_{j+1} < \ell_j$ $(j \geqq 0)$ and $\ell_0 \leq 1$. Let E be a symmetric generalized Cantor set constructed by $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$. We have $B_{\beta,q}(E) = 0$ by the Theorem, since $\sum_{j=1}^{\infty}(2^{nj}\ell_j^{n-\beta q})^{-1/(q-1)} = \infty$. Thus the set E is (β,q)-thin at every point. Next, we show that E is not (α,p)-thin at each of its points. Since $E \cap B(x, r)$ contains some symmetric generalized Cantor sets for every $x \in E$ and $r > 0$, by using the lower estimate of our theorem we can prove that

$$\int_0^1 (r^{-d}B_{\alpha,p}(E \cap B(x, r))^a r^{-1}dr = \infty.$$

The case (iv). Let $k_j = 2$ for $j \geqq 1$ and let $\ell_j = \exp\{-(j+j_0)^{-1}2^{n(j+j_0)/(q-1)}\}$ for $j \geqq 0$, where $j_0(\geqq 0)$ is so chosen that $2\ell_{j+1} < \ell_j$ for all $j \geqq 0$ and $n^{1/2}\ell_0 < 1$. Let E be a symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$. Then we can obtain $B_{\beta,q}(E) = 0$ and for every $x \in E$

$$\int_0^1 \{B_{\alpha,p}(E \cap B(x, r))\}^a r^{-1} dr = \infty.$$

Therefore in each case we have constructed a Cantor set with desired properties.

Finally, take a point $x_0 \in E$ and set $H = (R^n \setminus E) \cup \{x_0\}$. Then $H \in \mathcal{T}_{\beta,q} \setminus \mathcal{T}_{\alpha,p}$, because $B_{\beta,q}(E) = 0$, $R^n \setminus H = E \setminus \{x_0\}$ and $E \setminus \{x_0\}$ is not (α,p)-thin at x_0.

Remark. For the present I can not construct a set contained in $\mathcal{T}_{\alpha,p} \setminus \mathcal{T}_{\beta,q}$ in the case where $0 < \beta q \leqq \alpha p < n$ and $(n - \beta q)/(q - 1) < (n - \alpha p)/(p - 1)$ by using symmetric generalized Cantor sets.

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Faculty of Education,
Shimane University